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Convex Power Constructions for Continuous D-Cones

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Abstract

Powerdomains were introduced to describe non-deterministic behaviour of computational logic. Closely related are the Hoare, Smyth and Plotkin powerdomain while the probabilistic powerdomain is of a distinct different structure. Jones and Plotkin took the set of all valuations on a continuous domain to model probabilistic choice. Valuations can be added, and multiplied by non-negative real numbers. Thus, the probabilistic powerdomain satisfies the real cone properties in addition to being a continuous domain with respect to the pointwise order on valuations. Moreover, the order connects nicely to the algebraic operations, since addition and scalar multiplication turn out to be Scott-continuous. In general, a real cone of this type is called a continuous d-cone.

Various people asked the question, what happens if non-deterministic choice is combined with probabilistic choice. Can the known power constructions be combined somehow to get a suitable mathematical object, in which both kinds of non-determinism can be interpreted? McIver and Morgan tackle the problem for finite state spaces S. They consider a subset of the Plotkin powerdomain over the space of distributions on S. We construct something similar in the more general setting of continuous d-cones. We will introduce a Hoare and a Smyth type of construction for continuous d-cones. We also define addition and scalar multiplication on these powerdomains in such a way that they again become a continuous d-cone. It will turn out that these powercones are universal in a setting of continuous d-cones with an additional semilattice operation. It remains to work out a Plotkin type of construction for continuous d-cones. This should combine the features of the convex Hoare and Smyth powercones presented in this paper.

1 Introduction

Consider non-deterministic behaviour of a computer program. Naturally, some kind of powerset construction arises in modelling the different possible outcomes of the program for the same input value. The first semantic of such a feature within the framework of domain theory was given by Plotkin in [10].

Here, he introduces the Plotkin powerdomain to capture non-deterministic branching. Briefly after this, in [13] Smyth proposes a simpler, half sided powerdomain. This one describes a demonic view of non-determinism, while an angelic view is modelled in the Hoare powerdomain. The Plotkin powerdomain combines these two views of non-determinism. For all three powerdomains exist, under certain assumptions on the underlying space, nice topological characterisations as a special subset of the whole powerset. A general algebraic framework of these and other powerdomain constructions is studied by Heckmann in [3]. An approach via free continuous domain algebras is chosen by Abramsky and Jung in [1]. Models for a probabilistic view of non-determinism were introduced by Jones and Plotkin in [6,5]. The probabilistic powerdomain is given as the set of all continuous valuations from the underlying space into the unit interval $[0, 1]$. If one allows values in $\overline{\mathbb{R}}_+$ instead, the extended probabilistic powerdomain is obtained. It is shown by Kirch in [7] to be the free continuous d-cone over a continuous domain. A research group in Oxford wondered how from the known models new ones could be obtained which combine the non-deterministic view with the probabilistic one. From this group, in [9] Morgan and McIver tackle the problem for finite state spaces S and build a Plotkin style powerdomain over the space of probability distributions on S. We will construct a Hoare and a Smyth style construction for the extended probabilistic powerdomain in this paper. However, we will define these constructions for continuous d-cones in general. As future work remains to develop a Plotkin style power construction for continuous d-cones.

In section 2 we define continuous d-cones and our main example, the extended probabilistic powerdomain. We also recall useful properties of convex sets and various features which arise in the setting of continuous domains and Scott continuous functions. The convex Hoare powercone of a continuous d-cone is introduced in section 3. Similar to the topological characterisation of the classical Hoare powerdomain, it consists of the closed convex subsets of the original d-cone ordered by inclusion. Convexity enables us to lift addition and scalar multiplication to the convex Hoare powercone in such a way that it also becomes a continuous d-cone. However, there exist binary suprema as an extra semilattice operation. In this context the Hoare powercone turns out to be the free sup-d-cone over the original continuous d-cone. In section 4, exchanging in the topological characterisation of the Smyth powerdomain non-empty compact saturated subsets by convex such, only leads to a d-cone with extra binary infima. To receive the continuous inf-d-cone over a continuous d-cone, a further restriction to approximable subsets is necessary. This construction is possible for continuous d-cones with additive way-below relation only. This additional hypothesis is always fulfilled for the probabilistic powerdomain of a continuous domain.

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2 Basic Concepts

2.1 Continuous D-Cones

The basic notions of directed complete partial ordered sets, abreviated as dcpo's, and continuous domains will be taken from [1]. In this section we will introduce continuous d-cones and our main example, the extended probabilistic powerdomain.

Definition 2.1 A set C is called *real cone* if there exist $0 \in C$ and two operations $+\colon C \times C \to C$ and $\cdot\colon \mathbb{R}_+ \times C \to C$ such that for all $a, b, c \in C$ and for all $r, s \in \mathbb{R}_+$ the following properties are satisfied:

$$\begin{aligned}
(a+b)+c &= a+(b+c)\\
a+b &= b+a\\
a+0 &= a\\
1\cdot a &= a\\
0\cdot a &= 0\\
(r\cdot s)\cdot a &= r\cdot(s\cdot a)\\
r\cdot(a+b) &= (r\cdot a)+(r\cdot b)\\
(r+s)\cdot a &= (r\cdot a)+(s\cdot a)
\end{aligned}$$

A real cone C is called *d-cone* if C is also a dcpo such that addition and scalar multiplication are continuous with respect to the Scott topology on C and the lower topology on $\mathbb{R}_+$. In the case that C is a continuous domain then C is called a *continuous d-cone*.

Remark that on d-cones the scalar multiplication can be extended to ∞ by $\infty \cdot x := \bigvee^{\uparrow}_{r \geq 0} r \cdot x$. The real cone axioms will also hold for ∞ since all operations preserve directed suprema.

Throughout this paper, our main example will be the extended probabilistic powerdomain of a continuous domain. However, a definition can be given for arbitrary topological spaces:

Definition 2.2 Let X be a topological space, $\mathcal{O}(X)$ the open sets of X, and let $\overline{\mathbb{R}}_+$ be equipped with the Scott-topology.

A function $\mu\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ is called *continuous valuation on* X if, for all $U, V \in \mathcal{O}(X)$ and directed $(U_i)_{i \in I} \subseteq \mathcal{O}(X)$, it satisfies :

- strictness: $\mu(\emptyset) = 0$,
- monotonicity: $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$,
- modularity: $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$,
- Scott-continuity: $\mu(\bigcup^{\uparrow}_{i \in I} U_i) = \bigvee^{\uparrow}_{i \in I} \mu(U_i)$.

The set of all continuous valuations on X is denoted by $\mathcal{V}(X)$ and called the *extended probabilistic powerdomain of* X.

The set $\mathcal{V}(X)$ yields a d-cone with respect to the pointwise order, addition and scalar multiplication. This d-cone is continuous whenever the underlying space X is a continuous domain. As an additional feature, the way-below relation is additive on the extended probabilistic powerdomain of a continuous domain. Thereby, the way-below relation of a continuous d-cone C is called *additive* if whenever $a_1 \ll b_1$ and $a_2 \ll b_2$ in C then is also $a_1 + a_2 \ll b_1 + b_2$. A proof of this property of $\mathcal{V}(X)$ and other examples of continuous d-cones with additive way-below relation can be found in [15]. A similar property for scalar multiplication holds in any d-cone, namely $r \cdot a \ll r \cdot b$ for $a \ll b$ and $r \in \mathbb{R}_+$. These two characteristics of the way-below relation will turn out to be necessary for a suitable analogy of the Smyth powerconstruction for continuous d-cones.

2.2 Properties of Convexity

As usual, a subset A of a d-cone C is called *convex* if $r \cdot a + (1-r) \cdot b \in A$, for all $a, b \in A$ and $r \in [0,1]$. Note that there exists also a notion of order convexity which arises in the context of the Plotkin powerdomain, however is not meant in this paper by convexity. Next, we collect some observations about convex subsets of d-cones.

Lemma 2.3 *Let M be a convex subset of a d-cone. Then holds:*

(i) *The closure $\overline{M}$ is again convex.*

(ii) *The saturation $\uparrow M$ is also convex.*

Proof. A d-cone can be seen as a topological semivectorspace in the sense of Prakash and Sertel. Hence, their proof [11, Proposition 2.2] applies to our situation, and yields that the closure of a convex set is convex. Note that no Hausdorff space is needed.

That the saturation of a convex set is convex is an immediate consequence of the fact that addition and scalar multiplication of a d-cone are monotone. $\square$

Lemma 2.4 *For a d-cone with an additive way-below relation, the interior of any convex saturated set M is convex.*

Proof. Let $x, y \in \mathrm{i}nt\, M$. Then, there exist $x', y' \in M$ with $x' \ll x$ and $y' \ll y$. Points on the line connecting x and y are given by $rx + (1-r)y$ for $r \in [0,1]$. Using that the way-below relation is additive, we conclude $rx + (1-r)y \gg rx' + (1-r)y' \in M$. $\square$

Purely algebraic properties are

Lemma 2.5 *Let P, Q be subsets of a real cone. Then we have:*

(i) *The convex closure of the sum is given by* $\mathrm{conv}(P+Q) = \mathrm{conv}\, P + \mathrm{conv}\, Q$.

(ii) *The convex closure of the union is given by* $\mathrm{conv}(P \cup Q) = \{rp + (1-r)q \mid p \in P, q \in Q, r \in [0,1]\}$.

If we apply the second part of the previous lemma to two singleton sets $\{x\}$ and $\{y\}$ we see that the convex hull of the two element set $\{x, y\}$ is indeed the line connecting x and y. By a simple induction over the cardinality of a finite set F we conclude for the convex hull $conv\, F = \{\sum_{x\in F} r_x x \mid x \in F, r_x \in [0,1], \sum_{x\in F} r_x = 1\}$.

Throughout this paper compactness is to be understood as the property that for every covering with open sets there exists a finite subcovering.

Lemma 2.6 *For P and Q compact, $conv(P \cup Q)$ is also compact.*

Proof. We set $\Delta := \{(r, 1-r) \mid r \in [0,1]\}$. This set is compact with respect to the Scott topology on $[0,1]^2$. The map $conv\colon \Delta \times C \times C \to C$, which maps $((r, 1-r), x, y)$ to $rx + (1-r)y$ is continuous and the convex hull of $P \cup Q$ is equal to the image of the compact set $\Delta \times P \times Q$, thus also compact. □

We can apply this Lemma also to two singleton sets and get by induction over the cardinality of finite sets F that their convex closure $conv\, F$ is always compact.

2.3 Some Topological and Order Theoretical Facts

An elementary and useful topological fact is

Lemma 2.7 *Let X and Y be topological spaces, $f\colon X \to Y$ a continuous function. Then is $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A \in X$.*

Lemma 2.8 *In a continuous domain the Scott-closure of an arbitrary subset S is*

$$\{\bigvee{}^{\uparrow} S' \mid S' \text{ directed subset of } \downarrow S\}.$$

Proof. See e.g. [12, Lemma 6.4]. □

Lemma 2.9 *Let $f\colon X \to Y$ be a monotone map between dcpo's and S a subset of X. The join of $f(S)$ exists in Y iff the join of $f(\overline{S})$ exists in Y. In this case, $\bigvee f(S) = \bigvee f(\overline{S})$.*

Proof. This follows from [12, Corollary 1.6.(i)]. □

The *saturation* of any subset A in a topological space is defined to be the intersection of all the neighbourhoods of A. For a dcpo this yields that the saturation of A is equal to $\uparrow A$. As an immediate consequence from this definition we receive:

Lemma 2.10 *The saturation of a compact set K in a dcpo is compact.*

Lemma 2.11 *For a monotone map $f\colon X \to Y$ between dcpo's holds $\uparrow f(\uparrow A) = \uparrow f(A)$ for any subset A of X.*

Lemma 2.12 *In a continuous domain each compact subset has a neighbourhood basis of compact saturated sets.*

Proof. Let P be a compact subset of the continuous domain X and U an open neighbourhood of P. We can rewrite the open set as $U = \bigcup_{x \in U} \Uparrow x \supseteq P$. Since P is compact and the $\Uparrow x$ are open, finitely many already cover P. This means there exist $x_1, \ldots, x_n \in U$ with $P \subseteq \bigcup_{i=1}^n \Uparrow x_i \subseteq \bigcup_{i=1}^n \uparrow x_i \subseteq U$. The set $\bigcup_{i=1}^n \uparrow x_i$ is a compact neighbourhood of P contained in U. $\square$

Theorem 2.13 (Hofmann–Mislove) *For a sober space X the sets of open neighbourhoods of compact saturated sets are precisely the Scott-open filters on $\mathcal{O}(X)$.*

This result first appeared in [4]. A proof can also be found in [8]. A consequence which we will excessively use is

Corollary 2.14 *Let X be a sober space. The filtered intersection of a family of non-empty compact saturated subsets is compact and non-empty. If such a filtered intersection is contained in an open set O then some element of the family is contained in O already.*

The set of all compact saturated subsets of a topological space X ordered by reversed inclusion yields a dcpo, called $Q(X)$.

3 The Convex Hoare Powercone

3.1 The Convex Hoare Construction

For a continuous d-cone $(C, +, 0, \cdot)$ we define

$$\mathcal{C}(C) := \{A \subseteq C \mid A \neq \emptyset, \text{ convex, closed}\}, \text{ ordered by inclusion.}$$

The algebraic operations are lifted to $\mathcal{C}(C)$ in the following way

$$\boxplus\colon \mathcal{C}(C) \times \mathcal{C}(C) \to \mathcal{C}(C),\ A \boxplus B := \overline{A + B},$$

$$\boxdot\colon \mathbb{R}_+ \times \mathcal{C}(C) \to \mathcal{C}(C),\ r \boxdot A := r \cdot A,$$

where $\overline{A+B}$ is the topological closure of $A + B := \{a + b \mid a \in A, b \in B\}$, and $r \cdot A = \{r \cdot a \mid a \in A\}$. For these operations we can show that $(\mathcal{C}(C), \{0\}, \boxplus, \boxdot)$ is also a continuous d-cone. Moreover, there exist arbitrary suprema in $\mathcal{C}(C)$ and are preserved by addition and scalar multiplication . It will take the remainder of this section to prove this.

Immediately from the definition we get that

$$\{0\} \text{ will be the neutral element for addition in } \mathcal{C}(C).$$

Directed suprema are given by the topological closure of directed unions

$$\bigvee{}^{\uparrow} A_i = \overline{\bigcup{}^{\uparrow} A_i},$$

and binary suprema are formed as the closure of the convex hull of the union of two sets,

$$A \vee B = \overline{conv(A \cup B)}.$$

Proposition 3.1 *The above defined* $(\mathcal{C}(C), \boxplus, \{0\}, \boxdot)$ *is a real cone.*

Proof. First, let us show that the operations $\boxplus$ and $\boxdot$ are well-defined. For convex sets A and B the pointwise sum $A + B$ is also convex. Its topological closure is convex by Lemma 2.3. Multiplication of A by an $r \in \mathbb{R}_+$ is either $\{0\}$, the neutral element of $\mathcal{C}(C)$, for $r = 0$. For $r > 0$ multiplication by r is an order isomorphism of the original d-cone C, hence $r \boxdot A = r \cdot A$ is again convex and closed.

Most of the real cone properties are straightforward to check. For associativity of $\boxplus$ we need that addition on C is continuous, hence $\overline{\overline{A+B}+C} \subseteq \overline{(A+B)+C}$ by Lemma 2.7. Scalar multiplication is distributive since $\mathcal{C}(C)$ contains only convex sets. $\square$

Proposition 3.2 *For a continuous d-cone C, the set $\mathcal{C}(C)$ with inclusion as order is a continuous domain. Moreover, $\mathcal{C}(C)$ turns out to be a complete lattice.*

Proof. Directed suprema in $\mathcal{C}(C)$ are given by $\bigvee^{\uparrow} A_i = \overline{\bigcup^{\uparrow} A_i}$. The union of the A_i is again convex but not necessarily closed. Proposition 2.3 guarantees that the closure will stay convex. Binary suprema are formed as the closure of the convex hull of the union of two sets, $A \vee B = \overline{conv(A \cup B)}$. Since the neutral element $\{0\}$ is the least element of $\mathcal{C}(C)$ the supremum of the empty set exists. This proves that $\mathcal{C}(C)$ is a complete lattice.

To show continuity of $\mathcal{C}(C)$ we imitate the proof of Proposition 6.5 in [12]. First, we need that $x \ll y$ in C implies $\downarrow x \ll \downarrow y$ in $\mathcal{C}(C)$. Let $\downarrow y \subseteq \overline{\bigcup^{\uparrow} A_i}$ for any directed subset $(A_i)_{i \in I} \subseteq \mathcal{C}(C)$. Since $\overline{\bigcup^{\uparrow} A_i} = \{\bigvee^{\uparrow} S \mid S \text{ directed}, S \subseteq \bigcup^{\uparrow} A_i\}$ by 2.8, there exist a directed set $S \subseteq \bigcup^{\uparrow} A_i$ with $y \leq \bigvee^{\uparrow} S$. The definition of $x \ll y$ yields $x \leq s$ for one $s \in S$. Then there is an $i \in I$ with $s \in A_i$. This implies $\downarrow x \subseteq \downarrow s \subseteq A_i$, hence $\downarrow x \ll \downarrow y$. Continuity of C yields for each convex closed subset $A = \bigvee \{\downarrow d \mid \exists\, a \in A\ d \ll a\} = \overline{\bigcup \{\downarrow d \mid \exists\, a \in A\ d \ll a\}}$. We just proved that $\downarrow d \ll \downarrow a \subseteq A$. Thus, $A = \bigvee \twoheaddownarrow A$ and the set $\twoheaddownarrow A$ is always directed since finite suprema exist in $\mathcal{C}(C)$. This completes the proof that $\mathcal{C}(C)$ is continuous. $\square$

For an alternative proof of continuity of $\mathcal{C}(C)$, we can use the fact that for the original Hoare powerdomain it is already known that a continuous domain yields a continuous powerdomain. We define a continuous section retraction pair between the convex Hoare powercone $\mathcal{C}(C)$ and the Hoare powerdomain $\mathcal{H}(C) := \{A \subset C \mid A \neq \emptyset, \text{closed}\}$. In this case $\mathcal{C}(C)$ is continuous as the retract of a continuous domain [1, Theorem 3.1.4]. The retraction is defined in the obvious way $r \colon \mathcal{H}(C) \to \mathcal{C}(C)$, $r(A) := \overline{conv\, A}$. The section is the inclusion map $j \colon \mathcal{C}(C) \to \mathcal{H}(C)$, $j(B) := B$.

Proposition 3.3 *Addition $\boxplus$ and scalar multiplication $\boxdot$ on $\mathcal{C}(C)$ are continuous with respect to the Scott topology.*

Proof. Monotonicity of addition is clear from its definition. Thus, it remains to show $(\bigvee^{\uparrow} A_i) \boxplus B = \overline{\overline{\bigcup^{\uparrow} A_i} + B} \subseteq \overline{\bigcup^{\uparrow} \overline{A_i + B}} = \bigvee^{\uparrow}(A_i \boxplus B)$. For $x \in \overline{\bigcup^{\uparrow} A_i}$ exists a directed set $(a_k)_{k \in K} \subseteq \bigcup^{\uparrow} A_i$ with $x = \bigvee^{\uparrow} a_k$. Addition on C is Scott continuous, thus, we can calculate for $b \in B$,

$$x + b = \bigvee{}^{\uparrow} a_k + b = \bigvee{}^{\uparrow}(a_k + b) \in \overline{\bigcup{}^{\uparrow}(A_i + B)} \subseteq \overline{\bigcup{}^{\uparrow} \overline{A_i + B}}.$$

Since the right hand side is closed, $(\bigvee^{\uparrow} A_i) \boxplus B =\subseteq \bigvee^{\uparrow}(A_i \boxplus B)$ follows.

We use that scalar multiplication on C is continuous and calculate for $\mathcal{C}(C)$,

$$\begin{aligned} (\bigvee{}^{\uparrow} r_i) \boxdot A &= \{(\bigvee{}^{\uparrow} r_i) \cdot a \mid a \in A\} \\ &= \{\bigvee{}^{\uparrow}(r_i \cdot a) \mid a \in A\} \\ &= \overline{\bigcup{}^{\uparrow} r_i \cdot A} \\ &= \bigvee{}^{\uparrow}(r_i \boxdot A). \end{aligned}$$

□

This proposition concludes our proof that for a continuous d-cone C, the convex Hoare powercone $\mathcal{C}(C)$ is also a continuous d-cone.

Binary suprema of elements in $\mathcal{C}(C)$ like we introduced them in the proof of Proposition 3.2 distribute over the algebraic operations.

Proposition 3.4 *The connection between binary suprema in $\mathcal{C}(C)$, $A \vee B = \overline{conv(A \cup B)}$, and addition and scalar multiplication is that for $A, B, C \in \mathcal{C}(C)$ and $r \in \mathbb{R}_+$ holds*

$$\begin{aligned} A \boxplus (B \vee C) &= (A \boxplus B) \vee (A \boxplus C) \\ r \boxdot (A \vee B) &= r \boxdot A \vee r \boxdot B. \end{aligned}$$

Proof. From the definition of supremum and monotonicity of addition we get immediately $(A \boxplus B) \vee (A \boxplus C) \subseteq A \boxplus (B \vee C)$. To see the other inclusion, we apply Lemma 2.7 to addition and get

$$A \boxplus (B \vee C) = \overline{A + \overline{conv(B \cup C)}} \subseteq \overline{A + conv(B \cup C)}.$$

The set $(A \boxplus B) \vee (A \boxplus C)$ is closed. Thus, it suffices to show that any $x \in A + conv\,(B \cup C)$ is contained in $(A \boxplus B) \vee (A \boxplus C)$. Such an element can be written as $x = a + \sum_{i=1}^{n} r_i b_i + \sum_{j=1}^{m} s_j c_j$ where $a \in A$, $b_i \in B$, $c_j \in C$, $r_i, s_j \in [0,1]$ and $\sum_{i=1}^{n} r_i + \sum_{j=1}^{m} s_j = 1$. Using distributivity and $1 \cdot a = a$ we rewrite

$$x = \sum_{i=1}^{n} r_i(a + b_i) + \sum_{j=1}^{m} s_j(a + c_j) \in conv((A + B) \cup (A + C)),$$

and see $x \in \overline{conv(\overline{A+B} \cup \overline{A+C})} = (A \boxplus B) \vee (A \boxplus C)$.

We have $0 \boxdot (A \vee B) = \{0\} = \{0\} \vee \{0\} = (0 \boxdot A) \vee (0 \boxdot B)$. For $r > 0$, multiplication by r is an isomorphism and we conclude

$$\begin{aligned} r \boxdot (A \vee B) &= r \cdot \overline{conv(A \cup B)} = \overline{r \cdot conv(A \cup B)} \\ &= \overline{conv(r \cdot A \cup r \cdot B)} = (r \boxdot A) \vee (r \boxdot B). \end{aligned}$$

□

Together with Scott continuity of addition and scalar multiplication on $\mathcal{C}(C)$ the last proposition yields that arbitrary suprema are preserved by these operations. So far, we have proved in this section

Theorem 3.5 *Let $(C, +, 0, \cdot)$ be a continuous d-cone. Then $(\mathcal{C}(C), \{0\}, \boxplus, \boxdot)$ is also a continuous d-cone, called the convex Hoare powercone of C. Arbitrary suprema exist in $\mathcal{C}(C)$ and are preserved under addition $\boxplus$ and scalar multiplication $\boxdot$.*

3.2 Universal Property of the Convex Hoare Construction

Like the original Hoare powerdomain, the convex Hoare powercone can also be described by a universal property. The d-cone $\mathcal{C}(C)$ is the free continuous sup-d-cone over a continuous d-cone C. In this section we will specify and prove this statement.

Let us look at our construction in a categorical setting. The continuous d-cones are the objects in the category **CCONE** with the Scott continuous linear functions as morphisms. In the category $\mathbf{CCONE}^{\vee}$ we collect those continuous d-cones as objects, where in addition binary suprema exist and distribute over addition and scalar multiplication in the following way:

$$\begin{aligned} a + (b \vee c) &= (a + b) \vee (a + c) \\ r \cdot (a \vee b) &= r \cdot a \vee r \cdot b \end{aligned}$$

These two conditions and Scott continuity yield that arbitrary suprema are preserved by addition and scalar multiplication. The morphisms in the category $\mathbf{CCONE}^{\vee}$ are the linear suprema preserving maps.

Proposition 3.6 *$C \mapsto \mathcal{C}(C)$ can be extended to a functor $\mathcal{C}\colon \mathbf{CCONE} \to \mathbf{CCONE}^{\vee}$ by mapping a Scott continuous linear function $f\colon C \to D$ between continuous d-cones to a suprema preserving linear map between sup-d-cones $\mathcal{C}(f)\colon \mathcal{C}(C) \to \mathcal{C}(D)$ where $\mathcal{C}(f)(A) := \overline{f(A)}$.*

Proof. We already showed that $\mathcal{C}(C)$ is a continuous d-cone where arbitrary suprema exist and distribute over addition and scalar multiplication.

For a linear Scott continuous function $f\colon C \to D$ between continuous d-cones we have to show that $\mathcal{C}(f)\colon \mathcal{C}(C) \to \mathcal{C}(D)$ is linear, Scott continuous and preserves binary suprema. All these are short calculations where one repeatedly applies Lemma 2.7 to the continuous functions f, addition or scalar multiplication on C to get one inclusion. To receive the other inclusion one uses that topological closure is a hull operator.

To finish the proof that $\mathcal{C}$ is functorial, we calculate for any closed set $A \subseteq C$,

$$\mathcal{C}(\mathrm{id}_C)(A) = \overline{\mathrm{id}_C(A)} = \overline{A} = A = \mathrm{id}_{\mathcal{C}(C)}(A).$$

Regarding composition we apply Lemma 2.7 again and get

$$\mathcal{C}(g \circ f)(A) = \overline{g(f(A))} = \overline{g(\overline{f(A)})} = (\mathcal{C}(g) \circ \mathcal{C}(f))(A).$$

$\square$

Lemma 3.7 *Besides $\mathcal{C}\colon \mathbf{CCONE} \to \mathbf{CCONE}^{\vee}$ we have the forgetful functor $\mathcal{U}\colon \mathbf{CCONE}^{\vee} \to \mathbf{CCONE}$ in the other direction. In this situation, the mapping $\downarrow\colon \mathrm{I}D_{\mathbf{CCONE}} \to \mathcal{U} \circ \mathcal{C}$ is a natural transformation where for each continuous d-cone C the morphism $\downarrow_C\colon C \to \mathcal{U} \circ \mathcal{C}(C)$ maps an element $x \in C$ to its down closure $\downarrow x$.*

Proof. First, we show that for every continuous d-cone C the map $\downarrow_C$ is Scott continuous and linear, hence, a morphism between d-cones. Scott continuity is clear from $\downarrow (\bigvee^{\uparrow} x_i) = \overline{\bigcup^{\uparrow} \downarrow x_i} = \bigvee^{\uparrow}(\downarrow x_i)$. To show additivity we calculate

$$\downarrow (x + y) = \downarrow (\downarrow x + \downarrow y) = \overline{\downarrow x + \downarrow y} = \downarrow x \boxplus \downarrow y.$$

For scalars $r \in \mathbb{R}_+$ we get $\downarrow r \cdot x = r \cdot \downarrow x = r \boxdot \downarrow x$.

Thus, it is left to prove that for any Scott continuous linear function $f\colon C \to D$ between continuous d-cones the following diagram [1] commutes

$$\begin{tikzcd}
\mathrm{I}D(C) \arrow[r, "\downarrow_C"] \arrow[d, "f"'] & \mathcal{UC}(C) \arrow[d, "\mathcal{UC}(f)"] \\
\mathrm{I}D(D) \arrow[r, "\downarrow_D"] & \mathcal{UC}(D)
\end{tikzcd}$$

It is always $\downarrow_D \circ f(x) = \downarrow f(x) \subseteq \overline{f(\downarrow x)} = (\mathcal{U} \circ \mathcal{C}(f)) \circ \downarrow_C (x)$. That f is monotone implies $f(\downarrow x) \subseteq \downarrow f(x)$, and that $\downarrow f(x)$ is closed yields even $\overline{f(\downarrow x)} \subseteq \downarrow f(x)$, the other inequality. $\square$

Now, we can prove the following universal property

Theorem 3.8 *The functor $\mathcal{C}\colon \mathbf{CCONE} \to \mathbf{CCONE}^{\vee}$ is left adjoint to the forgetful functor $\mathcal{U}\colon \mathbf{CCONE}^{\vee} \to \mathbf{CCONE}$. In other words the universal*

[1] All diagrams in this paper are drawn with 'Paul Taylor's commutative diagrams'.

property given by the following diagram holds:

$$\begin{tikzcd}
\textbf{CCONE} & C \arrow[r, "\downarrow_C"] \arrow[dr, "\forall f"'] & \mathcal{UC}(C) \arrow[d, "\mathcal{U}(\hat{f})"] & \mathcal{C}(C) \arrow[d, "\exists! \hat{f}"'] & \textbf{CCONE}^{\vee} \\
& & \mathcal{U}(L) & L &
\end{tikzcd}$$

Proof. To make the diagram commute we want to have $\hat{f}(\downarrow x) = f(x)$. In order to get a suprema preserving map we are forced to set $\hat{f}(A) := \bigvee_{a \in A} f(a)$. Since f is monotone $\hat{f}(\downarrow x) := \bigvee_{a \in \downarrow x} f(a) \leq f(x)$. But since $x \in \downarrow x$ we also have $f(x) \leq \hat{f}(\downarrow x)$, which proves that the diagram commutes. The above argument also provides uniqueness of $\hat{f}$. It remains to show that our definition yields a linear, Scott continuous and binary suprema preserving map.

Recall that in L addition and scalar multiplication will preserve arbitrary suprema and not just directed ones. It is $\hat{f}(A \boxplus B) = \bigvee \{f(x) \mid x \in \overline{A + B}\}$. So, let $x \in \overline{A + B}$, i.e. there exist a directed subset $S \subseteq \downarrow (A+B)$ with $x = \bigvee^{\uparrow} S$ by Lemma 2.8. We use continuity and additivity of f and the definition of S to get

$$\begin{aligned}
f(x) &= \bigvee_{s \in S}{}^{\uparrow} f(s) \\
&\leq \bigvee \{f(a+b) \mid a \in A, b \in B\} \\
&= \bigvee \{f(a) + f(b) \mid a \in A, b \in B\} \\
&= \bigvee_{a \in A} f(a) + \bigvee_{b \in B} f(b) \\
&= \hat{f}(A) + \hat{f}(B).
\end{aligned}$$

Thus, $\hat{f}(A \boxplus B) \leq \hat{f}(A) + \hat{f}(B)$. The other inequality is

$$\begin{aligned}
\hat{f}(A) + \hat{f}(B) &= \bigvee \{f(a+b) \mid a \in A, b \in B\} \\
&\leq \bigvee \{f(x) \mid x \in \overline{A + B}\} = \hat{f}(A \boxplus B),
\end{aligned}$$

which shows additivity of $\hat{f}$. For any scalar $r \in \mathbb{R}_+$ we calculate

$$\begin{aligned}
\hat{f}(r \boxdot A) &= \bigvee_{b \in r \cdot A} f(b) = \bigvee \{f(r \cdot a) \mid b = r \cdot a \in r \cdot A\} \\
&= \bigvee_{a \in A} r \cdot f(a) = r \cdot \bigvee_{a \in A} f(a) = r \cdot \hat{f}(A).
\end{aligned}$$

Monotonicity of $\hat{f}$ follows directly from monotonicity of f. Now, let $\mathcal{S} \subseteq \mathcal{C}(C)$ be directed. Then

$$\hat{f}(\bigvee{}^{\uparrow}\mathcal{S}) = \hat{f}(\overline{\bigcup{}^{\uparrow}\mathcal{S}}) = \bigvee\{f(x) \mid x \in \overline{\bigcup{}^{\uparrow}\mathcal{S}}\}$$
$$= \bigvee\{f(x) \mid x \in \bigcup{}^{\uparrow}\mathcal{S}\}\text{ , by Lemma 2.9}$$
$$= \bigvee_{A\in\mathcal{S}}{}^{\uparrow}\bigvee_{a\in A} f(a) = \bigvee_{A\in\mathcal{S}}{}^{\uparrow}\hat{f}(A),$$

which proves $\hat{f}$ is Scott continuous. For $A, B \in \mathcal{C}(C)$ we get

$$\begin{aligned}
\hat{f}(A \vee B) &= \hat{f}(\overline{conv(A \cup B)}) = \bigvee\{f(x) \mid x \in \overline{conv(A \cup B)}\} \\
&= \bigvee\{f(x) \mid x \in conv(A \cup B)\}\text{ , by Lemma 2.9} \\
&= \bigvee\{f(r \cdot a + (1-r) \cdot b) \mid a \in A, b \in B, r \in [0,1]\} \\
&= \bigvee\{r \cdot f(a) + (1-r) \cdot f(b) \mid a \in A, b \in B, r \in [0,1]\} \\
&\leq \bigvee\{r \cdot (f(a) \vee f(b)) + (1-r) \cdot (f(a) \vee f(b)) \\
&\qquad \mid a \in A, b \in B, r \in [0,1]\} \\
&= \bigvee\{f(a) \vee f(b) \mid a \in A, b \in B\} = \bigvee_{a\in A} f(a) \vee \bigvee_{b\in B} f(b) \\
&= \hat{f}(A) \vee \hat{f}(B).
\end{aligned}$$

The inequality $\hat{f}(A) \vee \hat{f}(B) \leq \hat{f}(A \vee B)$ follows from monotonicity of $\hat{f}$. Thus, $\hat{f}$ preserves binary suprema, which completes the proof. $\square$

3.3 The Way-Below Relation on the Convex Hoare Powercone

The way-below relation on $\mathcal{C}(C)$ can be described in terms of the way-below relation on C.

Lemma 3.9 *We have $A \ll B$ in $\mathcal{C}(C)$ if and only if there exist a finite set E in C such that $A \subseteq \overline{conv\, E}$ and for all $e \in E$ there exist $b \in B$ with $e \ll b$ in C.*

Proof. First, note that $\overline{conv\, E} = \overline{conv \downarrow E}$. Further, for the 'if'-part note that $\overline{conv \downarrow E} \ll B$ by the proof of Proposition 3.2. For $A \subseteq \overline{conv\, E} \subseteq \overline{conv \downarrow E}$ we get $A \ll B$.

For the 'only if'-part recall that $B = \bigvee{}^{\uparrow}\{\overline{conv \downarrow E} = \overline{conv \bigcup_{e\in E} \downarrow e} \mid E \text{ finite}, \forall e \in E\ \exists b \in B\ e \ll b\}$. Then $A \ll B$ implies the existence of a finite set E for which $A \subseteq \overline{conv \downarrow E}$ holds and for all $e \in E$ there exist $b \in B$ with $e \ll b$ in C. $\square$

Proposition 3.10 *If the continuous d-cone C has an additive way-below relation then $\mathcal{C}(C)$ also does.*

Proof. Let $A_1 \ll B_1$ and $A_2 \ll B_2$ in $\mathcal{C}(C)$. By the above characterisation this means for $i = 1, 2$ there exist a finite set E_i such that $A_i \subseteq \overline{conv \downarrow E_i}$ and for all $e \in E_i$ there exist $b \in B_i$ with $e \ll b$. We claim that $E := E_1 + E_2$ is a finite set which is witness for $A_1 \boxplus A_2 \ll B_1 \boxplus B_2$.

Surely, E is finite. To see $A_1 \boxplus A_2 = \overline{A_1 + A_2} \subseteq \overline{conv \downarrow E}$ it suffices to show $A_1 + A_2 \subseteq \overline{conv \downarrow E}$ since the latter set is closed. We get

$$\begin{aligned} A_1 + A_2 &\subseteq \overline{conv\, E_1} + \overline{conv\, E_2} \\ &\subseteq \overline{(conv\, E_1) + (conv\, E_2)}, \text{ by continuity of addition and 2.7} \\ &= \overline{conv(E_1 + E_2)}, \text{ by Lemma 2.5} \\ &= \overline{conv\, E}. \end{aligned}$$

For $e = e_1 + e_2 \in E$ exist $b_1 \in B_1$ and $b_2 \in B_2$ with $e_1 \ll b_1$ and $e_2 \ll b_2$. Since the way-below relation on C is additive we conclude $e_1 + e_2 \ll b_1 + b_2 \in B_1 \boxplus B_2$. □

4 The Convex Smyth Powercone

4.1 The Convex Smyth Construction — a Naive Approach

For a continuous d-cone $(C, +, 0, \cdot)$ we define

$$\mathcal{S}(C) := \{P \subseteq C \mid P \neq \emptyset, \text{ compact, saturated, convex}\},$$

ordered by reversed inclusion. The algebraic operations are lifted to $\mathcal{S}(C)$ in the following way

$$\oplus\colon \mathcal{S}(C) \times \mathcal{S}(C) \to \mathcal{S}(C),\ P \oplus Q := \uparrow (P + Q)$$

$$\odot\colon \mathbb{R}_+ \times \mathcal{S}(C) \to \mathcal{S}(C),\ r \odot P := \uparrow (r \cdot P) = \begin{cases} \uparrow \{0\} = C, & \text{if } r = 0 \\ r \cdot P, & \text{if } r > 0. \end{cases}$$

For these operations we will prove that they make $\mathcal{S}(C)$ into a d-cone in which binary infima exist. It will turn out that

$$C \text{ is the neutral Element for addition on } \mathcal{S}(C),$$

directed suprema are given as filtered intersection

$$\bigvee_{i \in I}^{\uparrow} P_i = \bigcap_{i \in I}{}_{\downarrow} P_i,$$

and binary infima are formed as

$$P \wedge Q = \uparrow conv(P \cup Q).$$

Unfortunately, it is not known when this d-cone is continuous. But we will see later in this section, that we can restrict the convex Smyth construction to the subset of approximable elements in $\mathcal{S}(C)$, which is continuous. To prove this we start with

Proposition 4.1 *The above defined $(\mathcal{S}(C), \oplus, C, \odot)$ is a real cone.*

Proof. First, we show that the operations $\oplus$ and $\odot$ are well-defined. The sum of two compact sets is again compact since addition on C is continuous. The sum of any two convex sets is again convex. Taking the upper set $\uparrow (P+Q)$ preserves compactness and convexity. Surely, this set is also non-empty whenever P and Q are non-empty. Multiplication by $r = 0$ yields $0 \odot P = \uparrow \{0\} = C$, an non-empty compact saturated subset of C. Since multiplication by $r > 0$ is an order-isomorphism we have $r \odot P = r \cdot P$ and non-empty compact saturated subsets are mapped to a set alike.

That addition on $\mathcal{S}(C)$ is associative follows from Lemma 2.11 using that addition on C is monotone. Commutativity of addition is immediately inherited. The original cone C is the neutral element in $\mathcal{S}(C)$ since it contains the neutral element 0 of C and because addition is monotone on C. It is $1 \odot P = \uparrow 1 \cdot P = \uparrow P = P$ and $0 \odot P = \uparrow 0 \cdot P = \uparrow \{0\} = C$, the neutral Element in $\mathcal{S}(C)$. That $(r \cdot s) \odot P = r \odot (s \odot P)$ is an immediate consequence of a monotone scalar multiplication on C and Lemma 2.11. Similar does the monotonicity of addition and scalar multiplication together with Lemma 2.11 and the according law of distributivity on C imply $r \odot (P \oplus Q) = r \odot P \oplus r \odot Q$. For the other law of distributivity we need in addition that the set P is convex, hence for $r \neq s$ and $p_1, p_2 \in P$ is also $\frac{r}{r+s} \cdot p_1 + \frac{s}{r+s} \cdot p_2 \in P$. Then $(r+s) \odot P = r \odot P \oplus s \odot P$ holds. $\square$

Proposition 4.2 *Directed suprema exist in $\mathcal{S}(C)$, hence it is a dcpo with respect to the order of reversed inclusion. Moreover, addition and scalar multiplication are Scott continuous on $\mathcal{S}(C)$.*

Proof. For a directed set $(P_i)_{i \in I}$ of non-empty compact convex saturated subsets of C, $\bigcap_{\downarrow i \in I} P_i$ is again non-empty compact convex and saturated. It is the biggest such set contained in all the P_i, hence it is their supremum with respect to the order of reversed inclusion.

Addition $\oplus$ on $\mathcal{S}(C)$ is monotone since addition on C respects set inclusion and building the upper closure of a set is monotone. It follows immediately $(\bigvee^{\uparrow}_{i \in I}) \oplus Q \geq \bigvee^{\uparrow}_{i \in I}(P_i \oplus Q)$. Thus, it is left to show $(\bigvee^{\uparrow}_{i \in I} P_i) \oplus Q \leq \bigvee^{\uparrow}_{i \in I}(P_i \oplus Q)$ or equivalently $\uparrow ((\bigcap_{\downarrow i \in I} P_i) + Q) \supseteq \bigcap_{\downarrow i \in I} \uparrow (P_i + Q)$. Since $\uparrow ((\bigcap_{\downarrow i \in I} P_i) + Q)$ is saturated it is sufficient to proof for each open neighbourhood O with $(\bigcap_{\downarrow i \in I} P_i) + Q \subseteq O$ that also $\bigcap_{\downarrow i \in I} \uparrow (P_i + Q) \subseteq O$. Addition on C is continuous, hence the inverse image of O under addition $\{(x, y) \in C \times C \mid x + y \in O\}$ is an open neighbourhood of $(\bigcap_{\downarrow i \in I} P_i) \times Q$. We apply Lemma 2.12 to the compact saturated sets $\bigcap_{\downarrow i \in I} P_i$ and Q in order to get a neighbourhood basis of compact sets U respectively V, and rewrite the set inclusion as

$$\begin{aligned}(\bigcap_{i\in I}{}^{\downarrow} P_i)\times Q &= \bigcap_{\bigcap^{\downarrow}_{i\in I} P_i\subseteq U}{}^{\downarrow} U \times \bigcap_{Q\subseteq V}{}^{\downarrow} V = \\ &= \bigcap_{\bigcap^{\downarrow}_{i\in I} P_i\subseteq U,\ Q\subseteq V}{}^{\downarrow} (U\times V) \subseteq \{(x,y)\in C\times C \mid x+y\in O\}. \quad (1)\end{aligned}$$

In this situation we can apply Corollary 2.14 and receive that there already exist compact neighbourhoods U of $\bigcap^{\downarrow}_{i\in I} P_i$ and V of Q with $U\times V \subseteq \{(x,y)\in C\times C \mid x+y\in O\}$. We also apply Corollary 2.14 to $\bigcap^{\downarrow}_{i\in I} P_i \subseteq int\,U$ and get that there exist an $i\in I$ with $P_i\subseteq int\,U\subseteq U$. Thus, $P_i\times Q\subseteq U\times V\subseteq \{(x,y)\in C\times C\mid x+y\in O\}$. Equivalently, this can be written as $P_i+Q\subseteq O$. Then the saturation $\uparrow(P_i+Q)$ is also contained in O. From this we conclude $\bigcap^{\downarrow}_{i\in I}\uparrow(P_i+Q)\subseteq O$, which completes the proof that addition on $\mathcal{S}(C)$ is Scott continuous in its first argument. The same argument works to prove Scott continuity in the second argument. This already implies addition on $\mathcal{S}(C)$ is Scott continuous.

That scalar multiplication $\odot$ on $\mathcal{S}(C)$ is monotone follows directly from monotonicity of scalar multiplication on C. This implies $\bigvee^{\uparrow}_{i,j}(r_i\odot P_j)\leq (\bigvee^{\uparrow}_i r_i)\odot(\bigvee^{\uparrow}_j P_j)$. It is left to show the other inequality, which can be rewritten as $\bigcap^{\downarrow}_{i,j}(r_i\odot P_j)\subseteq(\bigvee^{\uparrow}_i r_i)\odot(\bigcap^{\downarrow}_j P_j)$. In the case that $\bigvee^{\uparrow}_i r_i = 0$ it is $r_i=0$ for all $i\in I$. Then both sides are equal to C, the neutral element of $\mathcal{S}(C)$. In the case that $\bigvee^{\uparrow}_i r_i>0$ we can assume also that $r_i>0$ for all $i\in I$. The rest of the argument is similar to the one for addition. As saturated set, $(\bigvee^{\uparrow}_i r_i)\odot(\bigcap^{\downarrow}_j P_j)$ can be written as the intersection of all open sets O in which it is contained. If we can show that $\bigcap^{\downarrow}_{i,j}(r_i\odot P_j)\subseteq O$ for all those open sets we are finished. Thus, let O be an open set with $(\bigvee^{\uparrow}_i r_i)\odot(\bigcap^{\downarrow}_j P_j)\subseteq O$. Then, we also have $(\uparrow\bigvee^{\uparrow}_i r_i)\cdot(\bigcap^{\downarrow}_j P_j)\subseteq O$. Scalar multiplication on C is continuous, hence, the inverse image of O under scalar multiplication $\{(r,x)\in\mathbb{R}_+\times C\mid r\cdot x\in O\}$ is open and contains $(\uparrow\bigvee^{\uparrow}_i r_i)\times(\bigcap^{\downarrow}_j P_j)=(\bigcap^{\downarrow} V)\times(\bigcap^{\downarrow} U)=\bigcap^{\downarrow}(U\times V)$, where U and V are compact saturated neighbourhoods of $\uparrow\bigvee^{\uparrow}_i r_i$ respectively $\bigcap^{\downarrow}_j P_j$ of which sufficiently many exist by Lemma 2.12. To this situation we can apply Corollary 2.14 and receive that there exist U and V with $V\times U\subseteq\{(r,x)\in\mathbb{R}_+\times C\mid r\cdot x\in O\}$ or equivalently $U\cdot V\subseteq O$. Since $\bigvee^{\uparrow} r_i\in int\,V$ there exist $i\in I$ with $r_i\in int\,V$. Corollary 2.14 also implies for $\bigcap^{\downarrow} P_j\subseteq int\,V$ that there exists $j\in J$ with $P_j\subseteq int\,U$. Thus, we get

$$\bigcap{}^{\downarrow} r_i\odot P_j\subseteq r_i\odot P_j = r_i\cdot P_j\subseteq U\cdot V\subseteq O,$$

which completes the proof. □

Now, let us have a look at binary infima on $\mathcal{S}(C)$:

Lemma 4.3 *On $\mathcal{S}(C)$ binary infima exist and are given by $P\wedge Q=\uparrow conv(P\cup Q)$. The following holds for these infima:*

$$
\begin{aligned}
P \oplus (Q \wedge R) &= (P \oplus Q) \wedge (P \oplus R) \\
r \odot (P \wedge Q) &= (r \odot P) \wedge (r \odot Q).
\end{aligned}
$$

Proof. By Lemma 2.6 and Lemma 2.10, $\uparrow conv(P \cup Q)$ is also compact. It is the smallest convex saturated set which contains P and Q, hence $\uparrow conv(P \cup Q) = P \wedge Q$.

For the interaction of addition and infimum we calculate

$$
\begin{aligned}
P \oplus (Q \wedge R) &= \uparrow (P + \uparrow (conv(Q \cup R)) \\
&= \uparrow (P + conv(Q \cup R)), \text{ apply 2.11 to addition on } C \\
&= \uparrow \{p + \lambda q + (1-\lambda) r \mid p \in P, q \in Q, r \in R, \lambda \in [0,1]\}, \text{ 2.6} \\
&= \uparrow \{\lambda(p+q) + (1-\lambda)(p+r) \\
&\quad \mid p \in P, q \in Q, r \in R, \lambda \in [0,1]\} \\
&= \uparrow conv((P+Q) \cup (P+R)), \text{ by Lemma 2.6} \\
&= \uparrow conv(\uparrow (P+Q) \cup \uparrow (P+R)), \text{ apply 2.11} \\
&= (P \oplus Q) \wedge (P \oplus R).
\end{aligned}
$$

The interaction of scalar multiplication and infimum can be seen as follows

$$
\begin{aligned}
r \odot (P \wedge Q) &= \uparrow (r \cdot \uparrow conv(P \cup Q)) \\
&= \uparrow (r \cdot conv(P \cup Q)), \text{ apply 2.11 to scalar mult. on } C \\
&= \uparrow (conv(r \cdot P \cup r \cdot Q)) \\
&= \uparrow conv(\uparrow r \cdot P \cup \uparrow r \cdot Q), \text{ apply 2.11 to convex hull oper.} \\
&= r \odot P \wedge r \odot Q.
\end{aligned}
$$

$\square$

So far, we have shown for the naive approach of the convex Smyth construction:

Proposition 4.4 *Let $(C, +, 0, \cdot)$ be a continuous d-cone. Then the structure $(\mathcal{S}(C), \oplus, C, \odot)$ is a d-cone. Moreover, there exist binary infima in $\mathcal{S}(C)$.*

Unfortunately, it is not known whether the convex Smyth powercone is again a continuous d-cone. For a continuous d-cone C with an additive way-below relation continuity of $\mathcal{S}(C)$ can be derived from several equivalent conditions. All of them are founded on the characterisation of the way-below relation on the set of all compact saturated sets $\mathcal{Q}(C)$ (see also [1, Proposition 4.2.15]):

Lemma 4.5 *Let P be a compact saturated subset of a continuous domain. Then is Q way-below P in the order of reversed inclusion if and only if there exist an open set O with $P \subseteq O \subseteq Q$.*

Since $\mathcal{S}(C)$ is a subset of the sets of all compact saturated subsets of a continuous d-cone C two elements within $\mathcal{S}(C)$ are still way-below each other if the previous condition is fulfilled. But there might be more elements way-below a given compact saturated convex subset where one can not put an open set in between.

Proposition 4.6 *Let P be a compact saturated convex subset of a continuous d-cone C with additive way-below relation. Then the following is equivalent:*

(i) *the subset P is the filtered intersection of its compact saturated convex neighbourhoods.*

(ii) *The subset P is the filtered intersection of sets of the form $\uparrow \mathrm{conv}\, F$ with F a finite set and $P \subseteq \twoheaduparrow \mathrm{conv}\, F$.*

(iii) *The subset P has a neighbourhood basis of open convex sets.*

(iv) *For any point $x \notin P$ there exists a linear Scott continuous functional into which separates x from P.*

If the above conditions hold for all $P \in \mathcal{S}(C)$, we receive:

5. *The d-cone $\mathcal{S}(C)$ is continuous.*

Proof. $(1) \implies (2)$: For any (compact) saturated convex neighbourhood U of P holds $P \subseteq \mathrm{int}\, U = \bigcup_{x \in U} \twoheaduparrow x$. Since P is compact finitely many $\twoheaduparrow x$ suffice to cover P. We collect these finitely many elements in the finite set $f \subset U$ and conclude $P \subseteq \twoheaduparrow F \subseteq \twoheaduparrow conv\, F \subseteq \uparrow conv\, F \subseteq U$ since U is convex.

$(2) \implies (1)$: The sets $\uparrow conv\, F$ are compact saturated convex neighbourhoods of P.

$(2) \implies (3)$: Using that the way-below relation on C is additive, we receive that the sets $\twoheaduparrow conv\, F$ form a neighbourhood basis of P consisting of open convex sets.

$(3) \implies (2)$: This proof is the same as for $(1) \implies (2)$, since compactness was not needed there.

$(3) \implies (4)$: For $x \notin P$ we find an open convex neighbourhood O of P with $x \notin O$, hence $\downarrow x \cap O = \emptyset$. By the Separation Theorem [15, Theorem 4.4] there exist a linear Scott continuous functional $\Lambda \colon C \to \overline{\mathbb{R}}_+$ which separates $\downarrow x$ and O, thus, it also separates x from P.

$(4) \implies (3)$: Let $x \notin P$, $\Lambda \colon C \to \overline{\mathbb{R}}_+$ linear and Scott continuous with $\Lambda(x) \leq c < \Lambda(P)$, $c \in \mathbb{R}_+$. Then $\Lambda^{-1}(]c, \infty])$ is an open convex neighbourhood of P and $x \notin \Lambda^{-1}(]c, \infty])$.

$(1) - (4) \implies (5)$: This follows from Lemma 4.5. $\square$

We will call elements in $\mathcal{S}(C)$ for which conditions $(1) - (4)$ are fulfilled *approximable subsets of C*. Most or the time we will use the formulation in condition (ii). Condition (5) then states, the d-cone $\mathcal{S}(C)$ is continuous if all its elements are approximable. Otherwise, we will see in the next section that the approximable subsets of C build a continuous d-cone with respect to the restricted operations.

4.2 The Convex Smyth Powercone of Approximable Sets

In this section we will only consider continuous d-cones with additive way-below relation. In this case we know that the interior of a convex set is also convex. For instance, let us take for a finitely generated compact saturated convex set $\uparrow conv\, F$, F finite, an element $b \ll a$ for each $a \in F$, and let us collect those elements b in the finite set G. From additivity of the way-below relation we receive that $\uparrow conv\, F \subseteq int(\uparrow conv\, G) = \Uparrow conv\, G$. Moreover, $\Uparrow conv\, G$ is an open set between $\uparrow conv\, F$ and $\uparrow conv\, G$, hence $\uparrow conv\, G \ll \uparrow conv\, F$. Furthermore, $\uparrow conv\, F$ can be recovered as the filtered intersection of all possible $\uparrow conv\, G$, where G is built like before:

Proposition 4.7 *For a finite set F in a continuous d-cone C with additive way-below relation is $\uparrow conv\, F = \bigcap_{\downarrow}\{\uparrow conv\, G \mid \forall a \in F\ \exists b \in G\ b \ll a\}$.*

Proof. Let $F = \{a_1, \ldots, a_n\}$ and $x \notin \uparrow conv\, F$. Thus, for every $(r_i)_{i=1}^n \in \Delta := \{(r_i)_{i=1}^n \in [0,1]^n \mid \sum_{i=1}^n r_i = 1\}$ is $x \not\geq \sum_{i=1}^n r_i a_i = \bigvee^{\uparrow}\{\sum_{i=1}^n r_i b_i \mid b_i \ll a_i,\ i = 1, \ldots, n\}$. This implies that for all $i = 1, \ldots, n$ there exist $b_i \ll a_i$ such that $x \not\geq \sum_{i=1}^n r_i b_i$. For each of these tuples $(b_i)_{i=1}^n$ is the map $(r_i)_{i=1}^n \mapsto \sum_{i=1}^n r_i b_i \colon [0,1]^n \to C$ continuous. Hence the inverse image of $C \setminus \downarrow x$ is open. The above argument tells us that all these open inverse images cover the compact set $\Delta \subseteq [0,1]^n$. Thus, this covering of Δ contains a finite subcovering. For the finitely many b_i's that go with this subcovering we take an upper bound $c_i \ll a_i$. Then $x \not\geq \sum_{i=1}^n r_i c_i$ for all $(r_i)_{i=1}^n \in \Delta$. This means for $G = \{c_1, \ldots, c_n\}$ is $x \notin \uparrow conv\, G$. □

This proposition tells us that at least the finitely generated elements in $\mathcal{S}(C)$ are equal to the directed supremum of those finitely generated elements way-below. This means that sets of the form $\uparrow conv\, F$, F finite are approximable in the sense of Proposition 4.6. We take these elements of $\mathcal{S}(C)$ as a basis in order to get a subset that is a continuous domain with respect to the induced order. We will call this the convex Smyth Powercone of approximable sets $\mathcal{A}(C) := \{P \in \mathcal{S}(C) \mid P \text{ approximable}\}$. From the last proof also follows that the characterisation of the way-below relation on $\mathcal{Q}(C)$ from Lemma 4.5 also holds on $\mathcal{A}(C)$. Taking approximable sets only solves the problem that we could not show that $\mathcal{S}(C)$ is continuous. Instead, we will have to prove now that all operations can be restricted to $\mathcal{A}(C)$ in order to make this set into a d-cone, too.

Lemma 4.8 *For $P, Q \in \mathcal{A}(C)$ and $r \in \mathbb{R}_+$ is $P \oplus Q, r \odot P \in \mathcal{A}(C)$.*

Proof. Let $P = \bigcap_{\downarrow}\{\uparrow conv\, F \mid P \subseteq \Uparrow conv\, F\}$ and $Q = \bigcap_{\downarrow}\{\uparrow conv\, G \mid Q \subseteq \Uparrow conv\, G\}$ with F and G finite subsets of C. Because the way-below relation on C is additive we get $P + Q \subseteq \Uparrow conv(F + G)$, where $F + G = \{a + b \mid a \in F, b \in G\}$ is a finite set. We claim that $P \oplus Q = \bigcap_{\downarrow}\{\uparrow conv(F + G) \mid P \subseteq \Uparrow conv\, F, Q \subseteq \Uparrow conv\, G\}$. Since addition $\oplus$ is Scott continuous on $\mathcal{S}(C)$ we know

$$\begin{aligned} P \oplus Q &= \bigcap_{\downarrow}\{\uparrow conv\, F \mid P \subseteq \Uparrow conv\, F\} \oplus \bigcap_{\downarrow}\{\uparrow conv\, G \mid Q \subseteq \Uparrow conv\, G\} \\ &= \bigcap_{\downarrow}(\{\uparrow conv\, F \mid P \subseteq \Uparrow conv\, F\} \oplus \{\uparrow conv\, G \mid Q \subseteq \Uparrow conv\, G\}) \\ &= \bigcap_{\downarrow}\{\uparrow (\uparrow conv\, F + \uparrow conv\, G) \mid P \subseteq \Uparrow conv\, F,\ Q \subseteq \Uparrow conv\, G\} \\ &= \bigcap_{\downarrow}\{\uparrow (conv\, F + conv\, G) \mid P \subseteq \Uparrow conv\, F,\ Q \subseteq \Uparrow conv\, G\},\ 2.11 \\ &= \bigcap_{\downarrow}\{\uparrow conv(F+G) \mid P \subseteq \Uparrow conv\, F,\ Q \subseteq \Uparrow conv\, G\},\ \text{by } 2.5 \end{aligned}$$

We have $0 \odot P = \uparrow \{0\} = C = \uparrow conv\{0\} \in \mathcal{A}(C)$. For $r > 0$, multiplication by r is an order isomorphism, hence $r \odot P = r \cdot P = \bigcap_{\downarrow}\{\uparrow conv(r \cdot F) \mid P \subseteq \Uparrow conv\, F\}$ and $r \odot P \in \mathcal{A}(C)$. □

The last proof also shows that the neutral element C for addition on $\mathcal{S}(C)$ is in $\mathcal{A}(C)$. Since the operations can be restricted to $\mathcal{A}(C)$ all the real cone properties are inherited form $\mathcal{S}(C)$. It is straightforward from the definition of $\mathcal{A}(C)$ that directed suprema exist and are the same that one would get in $\mathcal{S}(C)$. Thus, Scott continuity of addition and scalar multiplication will also be passed on to $\mathcal{A}(C)$. The proof of Lemma 4.8 also shows for the sum of two basis elements $(\uparrow conv\, F) \oplus (\uparrow conv\, G) = \uparrow conv(F+G)$. This implies that the way-below relation is additive on the basis of finitely generated sets $\uparrow conv\, F$ and hence on the whole d-cone $\mathcal{A}(C)$. We have seen so far

Proposition 4.9 *Let C be a continuous d-cone with additive way-below relation. The subset $\mathcal{A}(C)$ of $\mathcal{S}(C)$ is also a continuous d-cone with additive way-below relation with respect to the restricted order, addition and scalar multiplication. A basis is given by the sets $\uparrow$ conv F where F is a finite subset of C.*

The proof of Lemma 4.8 also shows for the sum of two basis elements $(\uparrow conv\, F) \oplus (\uparrow conv\, G) = \uparrow conv(F+G)$. This implies that the way-below relation is additive on the basis of finitely generated sets $\uparrow conv\, F$ and hence on the whole d-cone $\mathcal{A}(C)$.

As an additional structure on $\mathcal{S}(C)$ there exist binary infima, and they are preserved by addition and scalar multiplication. This additional feature of the naive approach of the convex Smyth construction also holds within $\mathcal{A}(C)$.

Lemma 4.10 *The infimum of two approximable sets is approximable.*

Proof. For $P, Q \in \mathcal{A}(C)$, $P \wedge Q = \uparrow conv(P \cup Q)$ is the infimum in $\mathcal{S}(C)$. We claim that for $P = \bigcap_{\downarrow} \uparrow conv\, F$ and $Q = \bigcap_{\downarrow} \uparrow conv\, G$, $P \wedge Q = \bigcap_{\downarrow} \uparrow conv(F \cup G)$. The set of all these $\uparrow conv(F \cup G)$ is directed since the sets $\uparrow conv\, F$ and $\uparrow conv\, G$ are directed. Clearly, $P \wedge Q = \uparrow conv(P \cup Q) \subseteq \bigcap_{\downarrow} \uparrow conv(F \cup G)$. To show the other inclusion let $\Delta := \{(r, 1-r) \mid r \in [0,1]\}$. Then Δ is compact, $conv\colon \Delta \times C \times C \to C$, $((r, 1-r), a, b) \mapsto r \cdot a + (1-r) \cdot b$ is continuous and $conv(\Delta \times P \times Q) = conv(P \cup Q)$. By definition, $\uparrow conv(P \cup Q) = \bigcap_{\downarrow}\{O \mid O \text{ open}, conv(P \cup Q) \subseteq O\}$. For O open with $conv(P \cup Q) \subseteq O$,

$\Delta \times P \times Q \subseteq conv^{-1}(O)$, which is also open. Since P and Q are approximable we have

$$
\begin{aligned}
&\Delta \times P \times Q \\
&= \Delta \times \bigcap_{\downarrow}\{\uparrow conv\, F \mid P \subseteq \Uparrow conv\, F\} \times \bigcap_{\downarrow}\{\uparrow conv\, G \mid Q \subseteq \Uparrow conv\, G\} \\
&= \bigcap_{\downarrow}\{\Delta \times \uparrow conv\, F \times \uparrow conv\, G \mid P \subseteq \Uparrow conv\, F, Q \subseteq \Uparrow conv\, G\} \\
&\subseteq conv^{-1}(O).
\end{aligned}
$$

Here we have a filtered intersection of compact saturated sets contained in an open set. By Corollary 2.14 there exist finite sets F and G with $\Delta \times \uparrow conv\, F \times \uparrow conv\, G \subseteq conv^{-1}(O)$ and $P \subseteq \Uparrow conv\, F$, $Q \subseteq \Uparrow conv\, G$. This condition is equivalent to $\uparrow conv(F \cup G) = conv(\Delta \times \uparrow conv\, F \times \uparrow conv\, G) \subseteq O$, and implies

$$\bigcap_{\downarrow} \uparrow conv(F \cup G) \subseteq \bigcap_{\downarrow}\{O \mid O \text{ open}, conv(P \cup Q) \subseteq O\} = P \wedge Q.$$

For F, G finite is $F \cup G$ also finite, hence $P \wedge Q \in \mathcal{A}(C)$. □

Proposition 4.11 *There are the following connections between the infimum on $\mathcal{A}(C)$ and the other operations*

$$
\begin{aligned}
P \oplus (Q \wedge R) &= (P \oplus Q) \wedge (P \oplus R) \\
r \odot (P \wedge Q) &= (r \odot P) \wedge (r \odot Q) \\
\bigvee_{i,j}{}^{\uparrow}(P_i \wedge Q_j) &= (\bigvee_{i}{}^{\uparrow} P_i) \wedge (\bigvee_{j}{}^{\uparrow} Q_j)
\end{aligned}
$$

Proof. The first two properties already hold in $\mathcal{S}(C)$ and are still valid in $\mathcal{A}(C)$ since all the operations are the restricted ones from $\mathcal{S}(C)$. Concerning the last condition it follows from the definition of infimum and directed supremum that $\bigvee^{\uparrow}_{i,j}(P_i \wedge Q_j) \leq (\bigvee^{\uparrow}_i P_i) \wedge (\bigvee^{\uparrow}_j Q_j)$ always holds. To show the other inequality we will need that in $\mathcal{A}(C)$ we deal with approximable sets only. In particular, elements of $\mathcal{A}(C)$ are equal to the intersection of all open convex sets in which they are contained. Thus,

$$
\begin{aligned}
(\bigvee_{i}{}^{\uparrow} P_i) \wedge (\bigvee_{j}{}^{\uparrow} Q_j) &= \uparrow conv(\bigcap_{i}{}_{\downarrow} P_i \cup \bigcap_{j}{}_{\downarrow} Q_j) = \\
&= \bigcap_{\downarrow}\{O \mid O \text{ open, convex, } \bigcap_{i}{}_{\downarrow} P_i \cup \bigcap_{j}{}_{\downarrow} Q_j \subseteq O\}. \quad (2)
\end{aligned}
$$

Let $\bigcap_{\downarrow i} P_i \cup \bigcap_{\downarrow j} Q_j \subseteq O$, where O is open and convex. By Corollary 2.14 exist $P_i \subseteq O$ and $Q_j \subseteq O$, hence $P_i \cup Q_j \subseteq O$, and since O is open and convex we also have $P_i \wedge Q_j = \uparrow conv(P_i \cup Q_j) \subseteq O$. Finally, we receive

$$
\begin{aligned}
\bigcap_{\downarrow i,j}(P_i \wedge Q_j) &\subseteq \bigcap_{\downarrow}\{O \mid O \text{ open, convex, } \bigcap_{\downarrow i} P_i \cup \bigcap_{\downarrow j} Q_j \subseteq O\} \\
&= (\bigvee^{\uparrow}_i P_i) \wedge (\bigvee^{\uparrow}_j Q_j)
\end{aligned}
$$

or equivalently, using that $\mathcal{A}(C)$ is ordered by reversed inclusion, $\bigvee^{\uparrow}_{i,j}(P_i \wedge Q_j) \geq (\bigvee^{\uparrow}_i P_i) \wedge (\bigvee^{\uparrow}_j Q_j)$. □

4.3 The Universal Property of $\mathcal{A}(C)$

The convex Smyth power construction led to a manageable result only in the case of a continuous d-cone with additive way-below relation where we restrict the convex powercone to the approximable subsets. Thus, we can expect a universal property only within this setting. The continuous d-cones with additive way-below relation are the object in the category $\mathbf{CCONE^a}$ with Scott continuous linear functions as morphisms. In the category $\mathbf{CCONE}^\wedge$ we collect those continuous d-cones with additive way below relation where in addition binary infima exist and satisfy the following properties:

$$\begin{aligned} a + (b \wedge c) &= (a+b) \wedge (a+c) \\ r \cdot (a \wedge b) &= (r \cdot a) \wedge (r \cdot b) \\ \bigvee_{i,j}{}^{\uparrow}(a_i \wedge b_j) &= (\bigvee_i{}^{\uparrow} a_i) \wedge (\bigvee_j{}^{\uparrow} b_j). \end{aligned}$$

Morphisms in $\mathbf{CCONE}^\wedge$ are linear Scott continuous and binary, hence non-empty finite, infima preserving maps.

Proposition 4.12 *$C \mapsto \mathcal{A}(C)$ can be extended to a functor $\mathcal{A}\colon \mathbf{CCONE^a} \to \mathbf{CCONE}^\wedge$ by mapping a Scott continuous linear function $f\colon C \to D$ to $\mathcal{A}(f)\colon \mathcal{A}(C) \to \mathcal{A}(D)$ with $\mathcal{A}(f)(P) :=\uparrow f(P)$.*

Proof. We have just seen that $\mathcal{A}(C)$ is a continuous d-cone with additive way-below relation where in addition binary infima exist and the above connection between $\oplus$, $\odot$, $\bigvee^{\uparrow}$ and $\wedge$ is fulfilled, whenever C is a continuous d-cone with an additive way-below relation.

Next, we show that for a linear Scott continuous map $f\colon C \to D$ between continuous d-cones with additive way-below relation $\mathcal{A}(f)\colon \mathcal{A}(C) \to \mathcal{A}(D)$ is linear, Scott continuous and preserves binary infima. For $P \in \mathcal{A}(C)$, $\uparrow f(P)$ is compact since P is compact and f continuous, it is saturated by definition and convex since P is convex and f linear. If $P = \bigcap_{\downarrow}\{\uparrow \mathit{conv}\, F \mid P \subseteq\uparrow \mathit{conv}\, F\}$ then $\uparrow f(P) = \bigcap_{\downarrow}\{\uparrow \mathit{conv}(f(F)) \mid P \subseteq\uparrow \mathit{conv}\, F\}$. This shows that $\mathcal{A}(f)(P) =\uparrow f(P)$ is also approximable, hence an element of $\mathcal{A}(D)$. Since f and building the saturation of a set are monotone so is $\mathcal{A}(f)$. Thus, it follows immediately

$$\mathcal{A}(f)(\bigcap\nolimits_{\downarrow} P_i) =\uparrow f(\bigcap\nolimits_{\downarrow} P_i) \subseteq \bigcap\nolimits_{\downarrow}\uparrow f(P_i) = \bigcap\nolimits_{\downarrow}\mathcal{A}(f)(P_i).$$

To show the other inclusion we use $\uparrow f(\bigcap_{\downarrow} P_i) = \bigcap_{\downarrow}\{O \mid O \text{ open}, f(\bigcap_{\downarrow} P_i) \subseteq O\}$. Let O be open with $f(\bigcap_{\downarrow} P_i) \subseteq O$. Then $\bigcap_{\downarrow} P_i \subseteq f^{-1}(O)$ and the latter set is open by continuity of f. To this situation we apply Corollary 2.14 and receive that there exist $P_i \subseteq f^{-1}(O)$, or equivalently $f(P_i) \subseteq O)$. Then, $\uparrow f(P_i) \subseteq O)$ and $\bigcap_{\downarrow} \uparrow f(P_i) \subseteq\uparrow f(\bigcap_{\downarrow} P_i)$ which completes the argument that $\mathcal{A}(C)(f)$ is Scott continuous. To see that $\mathcal{A}(f)$ is linear and preserves binary infima one uses that f is linear and repeatedly applies Lemma 2.11 the monotone functions f, addition and scalar multiplication on D. Overall, $\mathcal{A}(f)$ is a morphism in $\mathbf{CCONE}^\wedge$. To finish the proof that $\mathcal{A}$ is functorial, we calculate for any compact saturated approximable set $P \subseteq C$,

$$\mathcal{A}(\mathrm{id}_C)(P) = \uparrow \mathrm{id}_C(P) = \uparrow P = P = \mathrm{id}_{\mathcal{A}(C)}(P).$$

Regarding composition we apply Lemma 2.11 to g and get

$$\mathcal{A}(g \circ f)(P) = \uparrow g(f(P)) = \uparrow g(\uparrow f(P)) = (\mathcal{A}(g) \circ \mathcal{A}(f))(P).$$

$\square$

Lemma 4.13 *Besides $\mathcal{A}\colon \mathbf{CCONE^a} \to \mathbf{CCONE}^\wedge$ we have the forgetful functor $\mathcal{U}\colon \mathbf{CCONE}^\wedge \to \mathbf{CCONE^a}$ in the other direction. In this situation $\uparrow\colon \mathrm{I}D_{\mathbf{CCONE^a}} \to \mathcal{U} \circ \mathcal{A}$ is a natural transformation where for each continuous d-cone C with additive way-below relation the morphism $\uparrow_C\colon C \to \mathcal{U}\circ\mathcal{A}(C)$ maps an element $x \in C$ to its upwards closure $\uparrow x$.*

Proof. First, we show that for every continuous d-cone C with additive way-below relation the map $\uparrow_C$ is Scott continuous and linear, hence, a morphism between d-cones. For $x \leq y$ is $\uparrow_C x \supseteq \uparrow_C y$, hence $\uparrow_C$ is monotone. Now, we show $\uparrow_C (\bigvee^\uparrow x_i) = \bigcap_\downarrow \uparrow_C x_i$. Therefore, let $y \in \bigcap_\downarrow \uparrow_C x_i$, which is equivalent to $y \geq x_i$ for all $i \in I$, i.e. $u \geq \bigvee^\uparrow x_i$ or equivalently $y \in \uparrow_C (\bigvee^\uparrow x_i)$. Thus, the two sets above are equal as we claimed. To show linearity we calculate

$$\begin{aligned}\uparrow_C (x+y) &= \{z \mid z \geq x+y\} = \{z \mid z \geq z_x + z_y, z_x \geq x, z_y \geq y\} \\ &= \uparrow_C (\uparrow_C x + \uparrow_C y) = \uparrow_C x \oplus \uparrow_C y \\ \uparrow_C (r \cdot x) &= \uparrow_C (r \cdot \uparrow_C x) = r \odot \uparrow_C x.\end{aligned}$$

Thus, it is left to prove that for any Scott continuous linear function $f\colon C \to D$ between continuous d-cones with additive way-below relation the following diagram commutes

$$\begin{tikzcd}
\mathrm{I}D(C) \arrow[r, "\uparrow_C"] \arrow[d, "f"'] & \mathcal{U}\mathcal{A}(C) \arrow[d, "\mathcal{U}\mathcal{A}(f)"] \\
\mathrm{I}D(D) \arrow[r, "\uparrow_D"] & \mathcal{U}\mathcal{A}(D)
\end{tikzcd}$$

From the monotonicity of f follows immediately $\uparrow_D f(\uparrow_C x) = \uparrow_D f(x)$ for all elements $x \in C$. $\square$

Now, we can prove the following universal property

Theorem 4.14 *The functor $\mathcal{A}\colon \mathbf{CCONE^a} \to \mathbf{CCONE}^\wedge$ is left adjoint to the forgetful functor $\mathcal{U}\colon \mathbf{CCONE}^\wedge \to \mathbf{CCONE^a}$. In other words the universal property given by the following diagram holds:*

$$
\begin{tikzcd}
 & C \arrow[r, "\uparrow_C"] \arrow[dr, "\forall f"'] & \mathcal{U}\mathcal{A}(C) \arrow[d, "\mathcal{U}(\hat{f})"] & & \mathcal{A}(C) \arrow[d, "\exists!\hat{f}"'] & \\
\mathbf{CCONE^a} & & \mathcal{U}(I) & & I & \mathbf{CCONE}^\wedge
\end{tikzcd}
$$

Proof. We claim that the following function has the desired properties,

$$\hat{f}(P) := \bigvee{}^{\uparrow}\{\bigwedge_{x\in F} f(x) \mid P \gg\uparrow \mathit{conv}\, F,\ F \text{ finite}\}.$$

The expression on the right hand side is defined in I since finite infima and directed suprema exist. That the set of which we take the supremum is indeed directed is an immediate consequence of

$$f(y) \geq \bigwedge_{x\in F} f(x), \text{ for } y \in\uparrow \mathit{conv}\, F. \tag{3}$$

From this observation also follows $\hat{f}(\uparrow x) \leq f(x)$. The other inequality can be deduced from the fact that for $y \ll x$ also $\uparrow y \ll\uparrow x$. Thus, $\hat{f}(\uparrow x) = f(x)$, which means that the diagram commutes. Condition (3) implies that $\uparrow \mathit{conv}\, F \mapsto \bigwedge_{x\in F} f(x)$ is a monotone map on a basis of $\mathcal{A}(C)$ into the dcpo I. The way $\hat{f}$ is defined from this is exactly the procedure proposed in [1, Proposition 2.2.24] to receive a Scott continuous function. It remains to show that $\hat{f}$ is linear and preserves binary infima. For $r = 0$, $r \oplus P =\uparrow \{0\} = C \ll C$, hence $\hat{f}(0 \odot P) = f(0) = 0 = 0 \cdot \hat{f}(P)$. For $r > 0$ is $P \gg\uparrow \mathit{conv}\, F$ if and only if $r \odot P \gg\uparrow \mathit{conv}(r\cdot F)$. Since f is homogeneous and scalar multiplication Scott continuous it follows $\hat{f}(r \odot P) = r \cdot \hat{f}(P)$. Additivity of $\hat{f}$ follows from additivity of f using the facts that $P \oplus Q = \bigcap_{\downarrow}\{\uparrow \mathit{conv}(F + G) \mid P \gg\uparrow \mathit{conv}\, F,\ Q \gg\uparrow \mathit{conv}\, G\}$ and whenever $P \oplus Q \gg\uparrow \mathit{conv}\, H$ then by continuity of addition on C and compactness of P and Q there exist F, G finite with $P \gg\uparrow \mathit{conv}\, F$, $Q \gg\uparrow \mathit{conv}\, G$ and $(\uparrow \mathit{conv}\, F) \oplus (\uparrow \mathit{conv}\, G) \gg\uparrow \mathit{conv}\, F$. Monotonicity of $\hat{f}$ implies $\hat{f}(P \wedge Q) \leq \hat{f}(P) \wedge \hat{f}(Q)$. The other inequality follows from the fact that whenever $P \gg\uparrow \mathit{conv}\, F$, $Q \gg\uparrow \mathit{conv}\, G$ then $P \wedge Q \gg\uparrow \mathit{conv}(F \cup G)$, hence we conclude $\hat{f}(P \wedge Q) = \hat{f}(P) \wedge \hat{f}(Q)$. □

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